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A HYBRID METHOD FOR COMPUTING THE FLOW OF VISCOELASTIC FLUIDS

P. D. ARIEL

Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, T2N 1N4 Canada

SUMMARY

A hybrid method for computing the flow of viscoelastic and second-order fluids is presented. It combines the features of the finite difference technique and the shooting method. The method is accurate because it uses central differences. Its convergence is at least superlinear.

The method is applied to obtain the solutions to three problems of flow of Walters' B' fluid: (a) flow near a stagnation point, (b) flow over a stretching sheet and (c) flow near a rotating disk. Numerical results reveal some new characteristics of flows which are not easy to demonstrate using the perturbation technique.

KEY WORDS Viscoelastic fluid Finite differences Shooting method Stagnation point flow Stretching sheet Rotating disk

INTRODUCTION

The study of the flow problems of a class of non-Newtonian fluids, which has come to be recognized as elastico-viscous fluids, is not only important technologically, but is also challenging to engineers, applied mathematicians and simulationists who are interested in obtaining accurate solutions. There are two special categories of elastico-viscous fluids, second-order fluid and Walters' fluid, which have particularly attracted the attention of researchers during the last two decades.

The main difficulty which arises in the solution of the flow problems of these fluids is that the constitutive equations of viscoelastic fluids usually generate a higher-order derivative term in the momentum equations in comparison with the equations for Newtonian fluids. Because of the apparent non-availability of extra boundary conditions, researchers tend to develop a regular perturbation solution of the problem, taking the solution for the Newtonian fluid as the primary solution and the first-order perturbed solution as the secondary solution. A classical example of this technique is the analysis of the two-dimensional flow of Walters' B' fluid about a stagnation point given by Beard and Walters.[1]

The equations of motion governing the flow can be reduced to the following non-linear ordinary differential equation in f:

$$f''' + ff'' + 1 - f'^2 + k(ff^{\text{iv}} - 2f'f''' + f''^2) = 0, \tag{1}$$

with the boundary conditions

$$f(0) = f'(0) = 0, \qquad \lim_{\eta \to \infty} f'(\eta) = 1, \tag{2}$$

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where f is a non-dimensional measure of the streamfunction characterizing the velocity and k is a measure of the viscoelasticity of the fluid.

Beard and Walters[1] solved the problem defined above by assuming

$$f=f_0+kf_1. \tag{3}$$

It turns out that the differential equations for both f_0 and f_1 are of third order each, and since precisely three boundary conditions are given on both f_0 and f_1, the solutions of the differential equations for f_0 and f_1 can be obtained numerically using any standard integration routine.

The work of Beard and Walters[1] also attracted considerable attention because of an interesting phenomenon of the boundary layer: the velocity in the boundary layer exceeds the mainstream velocity. Rajeshwari and Rathna[2] solved the same problem using the Karman–Pohlhausen method of integral momentum equations, but they did not report the aforementioned phenomenon. Frater[3] suggested that the overshoot of velocity in the boundary layer might be due to seeking a regular perturbation solution of the problem in terms of k (see equation (3)). He gave a convincing example to support his argument, although his example was in a different context. Other researchers obtained approximate solutions of equations (1) and (2) using either the integral momentum equations[4] or the perturbation solution technique.[5,6]

Of late there have been attempts to get more accurate solutions of these equations using weighted residual methods (MWR). These methods have the advantage of not requiring the problem of numerical integration to be addressed. Further, a greater accuracy can be obtained, in principle, by choosing more trial functions. Of various methods belonging to this category, perhaps the simplest and most convenient is the collocation point method. Serth[7] obtained a solution of equations (1) and (2) by taking the set of Laguerre polynomials as the trial functions, though he also used the Chebyshev and Legendre polynomials. Unfortunately, in order to obtain sufficiently accurate solutions for a Non-newtonian fluid ($k \neq 0$), he had to increase the number of trial functions sharply with k. For example, for a Newtonian fluid the number of trial functions required for six-digit accuracy was five, but for a non-Newtonian fluid ($k=0{\cdot}2$) it rose to 32. Ng[8] used the technique of goal programming, which is more common in solving the problems occurring in operations research, to reduce the number of trial functions. However, since his results were dependent on the choice of collocation points, it was not very clear at what stage the process of adding another collocation point must be stopped. Neither Serth[7] nor Ng[8] gave the velocity profiles in their work, so the question of the velocity in the boundary layer overshooting its mainstream value remained unanswered.

It is highly desirable that a numerical technique based on finite differences be developed to obtain accurate solutions of the flow problems of elastico-viscous fluids, such as the ones characterized by equations (1) and (2). Serth[7] remarked that integration techniques such as Runge–Kutta or predictor–corrector methods would fail if they were to be applied to the above system of equations. Bhatnagar and Zago,[9] seeking the solution of the problem of the flow of a second-order fluid between rotating coaxial disks, gave a technique based on finite differences which did not use the idea of a regular perturbation expansion. Essentially, they shoved the higher-order derivative term arising due to visco-elasticity to the right-hand side of the finite difference equations and continued to iterate the resulting equations until they eventually converged to a solution. Their technique, although novel, had certain drawbacks. Firstly, it used either the forward or backward difference approximation for a derivative rather than the central difference, which reduced the accuracy of the solution. Secondly, it used an iterative scheme which was only linear. Finally, it contained several parameters which needed to be chosen properly to ensure the convergence of the iterative scheme.

In the present paper we suggest a hybrid method which is free from the above-mentioned drawbacks. It combines the features of the finite difference technique and the shooting method. It is accurate since it uses central differences or averages. Its convergence is superlinear or quadratic depending on whether the secant or Newton's method is used to locate the missing initial conditions. It does not require extra parameters for convergence. Finally, the results can be considerably improved by invoking Richardson's extrapolation, pushing the accuracy of the method to the order of h^4, h being the mesh size. The main ideas behind the method are illustrated with the problem of Beard and Walters,[1] but two more problems are considered: (i) flow of a viscoelastic fluid over a stretching sheet and (ii) flow of a viscoelastic fluid near a rotating disk. Using the regular perturbation technique, the solution of the former problem has been given by Rajagopal *et al.*,[10] while that of the latter has been given by Elliot.[11] The problem of flow over a stretching sheet also admits an exact solution which has recently been noted by Troy *et al.*[12]

It is proposed to apply the technique reported in this paper to compute the solution of the flow problems of second-order and Walters' B' fluids between porous plates, disks, etc.

Since all the problems considered in this paper relate to Walters' B' fluid, it would be appropriate to give here the full equations of motion. The constitutive equation for Walters' B' fluid is

$$p_{ik} = -pg_{ik} + \tau_{ik}, \tag{4}$$

where p_{ik} is the stress tensor, p is the isotropic pressure, g_{ik} is the metric tensor of a fixed co-ordinate system x^i, and τ_{ik} for fluids with short memories (i.e. short relaxation times) can be written as

$$\tau^{ik} = 2\eta_0 d^{ik} - 2k_0 \bar{\tau}^{ik}. \tag{5}$$

In equation (5) d^{ik} is the rate-of-strain tensor defined by

$$d^{ik} = \tfrac{1}{2}(v^i_{,k} + v^k_{,i}), \tag{6}$$

and $\bar{\tau}^{ij}$ is given by

$$\bar{\tau}^{ij} = \frac{\partial d^{ij}}{\partial t} + d^{ij}_{,k} v^k - d^{ik} v^j_{,k} - d^{kj} v^i_{,k} + d^{ij} v^k_{,k}. \tag{7}$$

Finally, η_0 and k_0 are the limiting viscosity at small rate of shear and the short-memory coefficient respectively, defined by

$$\eta_0 = \int_0^\infty N(\tau)\,\mathrm{d}\tau, \tag{8}$$

$$k_0 = \int_0^\infty \tau N(\tau)\,\mathrm{d}\tau, \tag{9}$$

where $N(\tau)$ is the distribution function of relaxation time τ.

In deriving equation (5), the terms involving

$$\int_0^\infty \tau^n N(\tau)\,\mathrm{d}\tau \quad (n \geqslant 2)$$

have been neglected.

Using the equations of state (5) and (6), the equations of motion can be written in the form

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}\right)=-\nabla p+\eta_0\nabla^2\mathbf{v}-k_0\left(\frac{\partial}{\partial t}\nabla^2\mathbf{v}+2(\mathbf{v}\cdot\nabla)\nabla^2\mathbf{v}-\nabla^2[(\mathbf{v}\cdot\nabla)\mathbf{v}]\right), \tag{10}$$

$$\nabla\cdot\mathbf{v}=0. \tag{11}$$

TWO-DIMENSIONAL FLOW NEAR A STAGNATION POINT

In this section we consider the two-dimensional flow near a stagnation point (see Figure 1).

For a steady two-dimensional motion with velocity components

$$u=u(x, y), \qquad v=v(x, y), \qquad w=0 \tag{12}$$

the equation of motion (10) becomes

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\nabla^2 u-k_0^*\left\{\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)\nabla^2 u-\frac{\partial u}{\partial x}\nabla^2 u-\frac{\partial u}{\partial y}\nabla^2 v\right.$$
$$\left.-2\left[\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}+\frac{\partial v}{\partial y}\frac{\partial^2 u}{\partial y^2}+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\frac{\partial^2 u}{\partial x\partial y}\right]\right\}, \tag{13}$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\nabla^2 v-k_0^*\left\{\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)\nabla^2 v-\frac{\partial v}{\partial x}\nabla^2 u-\frac{\partial v}{\partial y}\nabla^2 v\right.$$
$$\left.-2\left[\frac{\partial u}{\partial x}\frac{\partial^2 v}{\partial x^2}+\frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial y^2}+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\frac{\partial^2 v}{\partial x\partial y}\right]\right\}, \tag{14}$$

where $\nu=\eta_0/\rho$ and $k_0^*=k_0/\rho$.

The equation of continuity becomes

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0. \tag{15}$$

Making the usual boundary layer approximations of viscous flow theory—namely, within the boundary layer u, $\partial u/\partial x$, $\partial^2 u/\partial x^2$ and $\partial p/\partial x$ are $O(1)$, y and v are $O(\delta)$, δ being the boundary layer

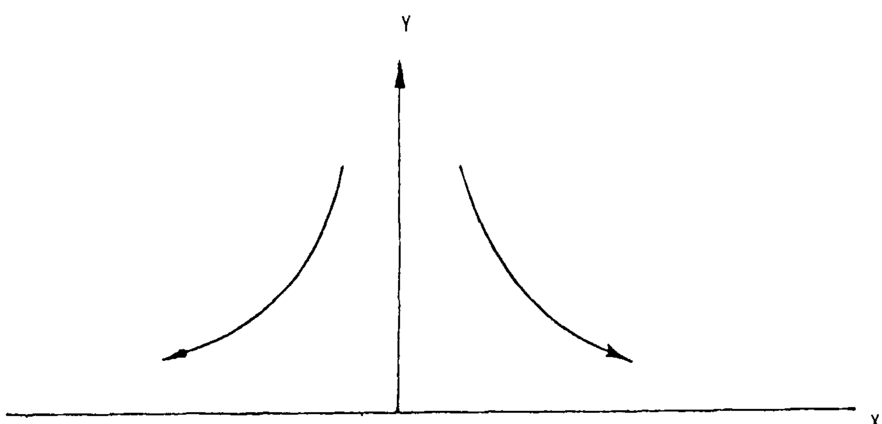

Figure 1. Geometry of flow near a stagnation point

thickness, and v and k_0^* are $O(\delta^2)$—equation (13) reduces to

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=U\frac{\mathrm{d}U}{\mathrm{d}x}+\nu\frac{\partial^2 u}{\partial y^2}-k_0^*\left(u\frac{\partial^3 u}{\partial x\partial y^2}+v\frac{\partial^3 u}{\partial y^3}+\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2}-\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x\partial y}\right), \tag{16}$$

where $U(x)$ is the mainstream velocity, which for the flow near a stagnation point is given by

$$U(x)=cx, \tag{17}$$

c being a constant.

We now introduce a streamfunction ψ given by

$$u=\frac{\partial\psi}{\partial y},\qquad v=-\frac{\partial\psi}{\partial x}. \tag{18}$$

With this choice of ψ the equation of continuity (15) is automatically satisfied.

It is now possible to define a similarity variable η as

$$\eta=\left(\frac{c}{\nu}\right)^{1/2}y, \tag{19}$$

such that if

$$\psi=(\nu c)^{1/2}f(\eta), \tag{20}$$

equation (16) is transformed to equation (1) with

$$k=k_0^*c/\nu. \tag{21}$$

The boundary conditions

$$u=0\quad\text{and}\quad v=0\quad\text{at } y=0,\qquad u\to U(x)\quad\text{as } y\to\infty \tag{22}$$

transform to equation (2).

Davies [4] has suggested imposing the condition

$$f'''(0)=-[1+kf''^2(0)] \tag{23}$$

to get the extra boundary condition required to solve equation (1). This condition essentially implies that as $k\to 0$, the solution must approach the Newtonian solution.

However, we note that we just need the boundedness of all the derivatives of f at $\eta=0$ to get the required solution of equation (1) subject to the boundary conditions (2). Thus, if one knows $f''(0)$, it should be possible to get a Taylor series expansion of f around $\eta=0$ from equation (1), notwithstanding the higher-order derivative occurring in the equation in the k-term. This idea suggests the following scheme.

Let

$$y_1=f,\qquad y_2=f',\qquad y_3=f'',\qquad y_4=f'''; \tag{24}$$

then equation (1) can be rewritten as

$$ky_1y_4'=-y_4-y_1y_3-1+y_2^2+k(2y_2y_3'-y_3^2)=0 \tag{25}$$

and the boundary conditions (2) and (23) as

$$y_1(0)=0,\qquad y_2(0)=0,\qquad y_4(0)=-[1+ky_3^2(0)],\quad \lim_{\eta\to\infty}y_2(\eta)=1. \tag{26}$$

Now, by assuming

$$f''(0)=s, \tag{27}$$

one can treat equation (25) as an initial value problem and iterate on s until the terminal asymptotic condition in (26) is satisfied. Indeed, this is how Teipel[13] attempted to get the solution of the corresponding problem for a second-order fluid. Simple though this scheme appears to be, it has certain drawbacks.

Firstly, note that there is a singularity at $\eta=0$ in the ODE (25), which means that in order to start the numerical integration by one of the standard integration routines, e.g. the Runge–Kutta method, one has to employ some other technique for obtaining the solution up to some value of η, say η_c. Since $f(\eta)$ is $O(\eta^2)$ at $\eta=0$, the singularity is quite strong at the origin. Therefore the value of η_c from where the Runge–Kutta method is to be applied must be sufficiently large to avoid problems of numerical instability. As a result, if one employs the Taylor series expansion to obtain the solution in $0\leqslant\eta\leqslant\eta_c$, a large number of terms will be needed to match the accuracy demanded by the Runge–Kutta method. Teipel[13] took the expansion up to the ninth derivative in order to obtain the required accuracy. This can become quite unwieldy for more complicated problems.

There is, however, a second drawback which is even more serious. Since the leading term y_4' of the ODE (25), besides being multiplied by y_1, which is $O(\eta^2)$, is further multiplied by k, the initial value problem can become highly unstable numerically for $k\to 0$. Thus it may become extremely difficult to get the solution for fluids which are slightly non-Newtonian.

With these drawbacks in view we now endeavour to develop a technique which is applicable for all values of k for which a solution is admissible, *including* the case $k=0$. We rewrite equation (1) as

$$y_3'+y_1y_3+1-y_2^2+k(y_1y_3''-2y_2y_3'+y_3^2)=0. \tag{28}$$

Let us introduce a mesh defined by

$$\eta_i=ih, \quad i=0, 1, 2, \ldots, N, \tag{29}$$

where N is a sufficiently large number.

It now seems natural to replace the first and second derivatives in equation (28) by the following formulae:

$$F'=\frac{F^{j+1}-F^j}{h}, \qquad F''=\frac{F^{j+1}-2F^j+F^{j-1}}{h^2}. \tag{30}$$

This leads to the system of equations

$$\frac{y_3^{j+1}-y_3^j}{h}+y_1^jy_3^j+1-(y_2^j)^2+k\left(y_1^j\frac{y_3^{j+1}-2y_3^j+y_3^{j-1}}{h^2}-2y_2^j\frac{y_3^{j+1}-y_3^j}{h}+(y_3^j)^2\right)=0, \tag{31}$$

$$\frac{y_2^{j+1}-y_2^j}{h}=y_3^j, \tag{32}$$

$$\frac{y_1^{j+1}-y_1^j}{h}=y_2^j. \tag{33}$$

The boundary conditions (2) become

$$y_1^0=0, \qquad y_2^0=0, \qquad y_2^N=1. \tag{34}$$

One can see from equation (31) that y_3^1 can be found from it if y_3^0 is known *regardless* of whether k is zero or not, because $y_1^0=0$. (Note that one does not need the value of y_3^{-1} because it is multiplied by y_1^0 which is zero.) This is the basic idea of the present method. The higher-order derivative occurring in the coefficient of k in equation (1) does *not* hinder the development of the solution at all.

To construct the rest of the solution, we proceed as follows. We obtain y_2^1 and y_1^1 from equations (32) and (33) respectively. At the next mesh point, since y_3^0 and y_3^1 are known, y_3^2 can be calculated from equation (31) and y_2^2 and y_1^2 from equations (32) and (33) respectively. Continuing in this manner, y_1^j, y_2^j and y_3^j can be computed at each mesh point.

However, in practice y_3^0 is not known. It must be chosen so that the asymptotic boundary condition $y_2^N=1$ is satisfied. Thus the problem now reduces to finding an appropriate y_3^0 for which $y_2^N=1$. One can use the shooting method in conjunction with the secant method or some other zero-finding algorithm to precisely locate this value of y_3^0.

The procedure given above is the simplest version of the method. Since we have used the forward difference formula to approximate the first derivatives, the accuracy of the method is only of order h. This can be improved to $O(h^2)$ if we use the central difference formula and the averages.

Using the approximation

$$F'=\frac{F^{j+1}-F^{j-1}}{2h}, \tag{35}$$

we rewrite equation (28) as

$$\frac{y_3^{j+1}-y_3^{j-1}}{2h}+y_1^j y_3^j+1-(y_2^j)^2+k\left(y_1^j\frac{y_3^{j+1}-2y_3^j+y_3^{j-1}}{h^2}-2y_2^j\frac{y_3^{j+1}-y_3^{j-1}}{2h}+(y_3^j)^2\right)=0. \tag{36}$$

Equation (36) can be explicitly solved for y_3^{j+1}. We have

$$y_3^{j+1}=\left(1+\frac{2k}{h}y_1^j-2ky_2^j\right)^{-1}\Big[y_3^{j-1}-2h[y_1^j y_3^j+1-(y_2^j)^2]$$
$$-k\left(2y_1^j\frac{-2y_3^j+y_3^{j-1}}{h}+2y_2^j y_3^{j-1}+2h(y_3^j)^2\right)\Big]. \tag{37}$$

Equations (32) and (33) are also replaced by the more accurate approximations

$$y_2^{j+1}=y_2^j+\tfrac{1}{2}h(y_3^j+y_3^{j+1}), \tag{38}$$

$$y_1^{j+1}=y_1^j+\tfrac{1}{2}h(y_2^j+y_2^{j+1}). \tag{39}$$

Since we have replaced the first derivatives in equation (28) by differences involving the values of y_3 at two non-adjacent mesh points, we have a situation similar to that encountered in using the midpoint formula

$$y_{n+1}=y_{n-1}+2hf(x_n, y_n) \tag{40}$$

for the solution of the IVP

$$\frac{dy}{dx}=f(x, y), \qquad y(x_0)=y_0. \tag{41}$$

Here one must know the value of y_1 to start the algorithm (40). This difficulty is resolved by finding its value by some other means.

For the algorithm under consideration we obtain the value of y_3^1 within an accuracy $O(h^2)$ by expanding y_3^1 in a Taylor series expansion around $\eta=0$. We have

$$y_3^1=y_3^0+h(y_3^0)'+\frac{h^2}{2!}(y_3^0)''+\dots \quad (42)$$

Replacing y_3^0 by $f''(0)$, we have

$$y_3^1=f''(0)+hf'''(0)+\frac{h^2}{2!}f^{\text{iv}}(0)+\dots \quad (43)$$

If $f''(0)=s$ as before, then $f'''(0)$ can be determined in terms of s from equation (23). To obtain the value of $f^{\text{iv}}(0)$, we differentiate equation (1) and set $\eta=0$. We have

$$f^{\text{iv}}(0)=0. \quad (44)$$

Therefore

$$y_3^1=s-h(1+ks^2). \quad (45)$$

Now the integration proceeds as follows. First an approximate value of $f''(0)$ is chosen and y_3^1 is calculated from equation (45). Then y_2^1 and y_1^1 are obtained from equations (38) and (39) in that order. At the next cycle, since the values of y_3^0 and y_3^1 have become available, y_3^2 is calculated from equation (37). Then y_2^2 and y_1^2 are calculated from equations (38) and (39). The cycle is repeated until the values of y_1, y_2 and y_3 have been calculated at all the mesh points. Note that these values are computed in a specific order. First y_3 is computed, then y_2 and finally y_1. Once again a zero-finding algorithm can be chosen to pinpoint the value of s which would lead to $y_2^N=1$.

RESULTS AND DISCUSSION

The algorithm described by equations (37)–(39), with equation (45) providing the starting value of y_3, was translated into a FORTRAN program which was then run on the Honeywell Multics at the University of Calgary. The program incorporated the feature of improving the accuracy of the solution by invoking Richardson's extrapolation. First a few trial values of $f''(0)$ were chosen to estimate the value of $f'(\infty)$. Once a reasonable value had been found, the secant method was used to refine the value of $f''(0)$ iteratively. The iterations were stopped when two values of $f''(0)$ differed by less than 10^{-10}. For the refinement of the solution the stepsize was halved and a new solution obtained. For this solution the previously obtained value of $f''(0)$ turned out to be a reasonably good starting value. Finally, Richardson's extrapolation was used to get more accurate values of f, f' and f'' at the mesh points.

The program was initially run for values of $k=0$, 0·1, 0·2 and 0·3. It took 33 s to get all the results. In Figure 2, f is plotted against η for the above values of k. For the same values of k, f' is plotted against η in Figure 3. The most interesting feature of these results is that the velocity in the boundary layer oscillates about its value in the mainstream for sufficiently large values of η when $k\neq 0$, i.e. for the viscoelastic fluids, though the oscillations damp out with increasing η. This phenomenon is more pronounced for higher values of k (e.g. $k=0{\cdot}3$), but it was also observed for low values of $k(k=0{\cdot}05)$. A similar observation was also reported by Teipel[13] for the flow of a second-order fluid near a stagnation point. These results thus vindicate the conclusions of Beard and Walters.[1]

An attempt was made to find the range of values of k for which it was possible to obtain the solution by the present method. It was discovered that there existed a critical value of k, say k_c,

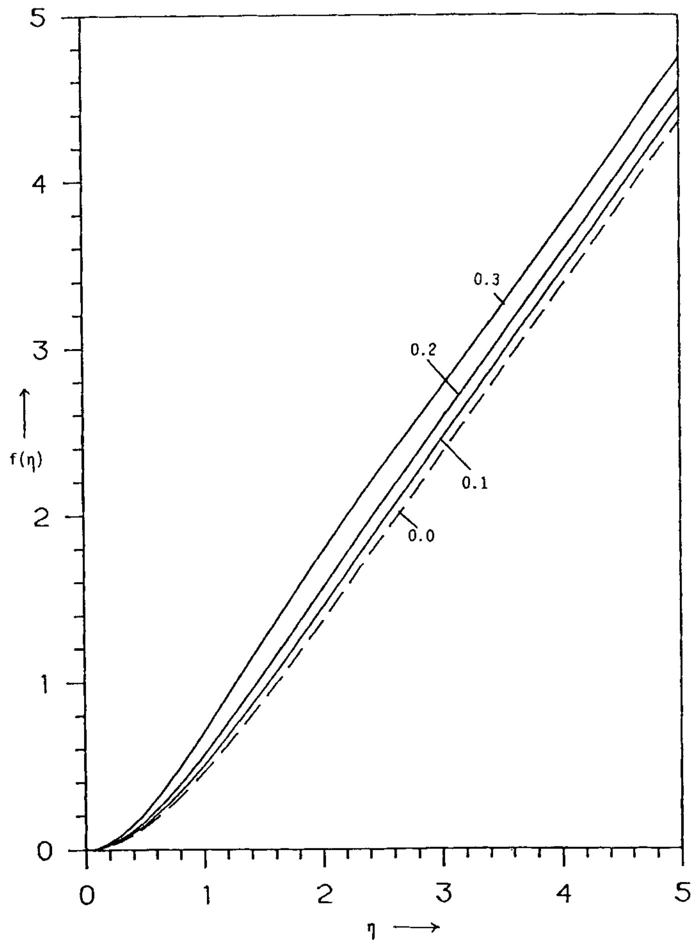

Figure 2. Variation of f with η for various values of k. Flow near a stagnation point

beyond which no solutions could be found. To see if this value of k_c was a turning point of the solution, the roles of k and s $(=f''(0))$ were reversed in seeking the solution, i.e. rather than finding s for a given k, k was obtained for a given value of s; s was increased monotonically from its starting value of 1·232 588 ($f''(0)$ for $k=0$) and the corresponding values of k calculated using the present method. The graph of $f''(0)$ against k is shown in Figure 4. From the results generated it was found that

$$k_c = 0{\cdot}325\,7864$$

For $k > k_c$ no solution could be found, but it appears that dual solutions exist for $k < k_c$ for all non-zero values of k. It is not necessary that both of these solutions are stable. It looks quite likely that the solutions on the upper branch of the curve in Figure 4 are unstable. However, one must undertake a stability analysis to ascertain the nature of these solutions. This has not been attempted here.

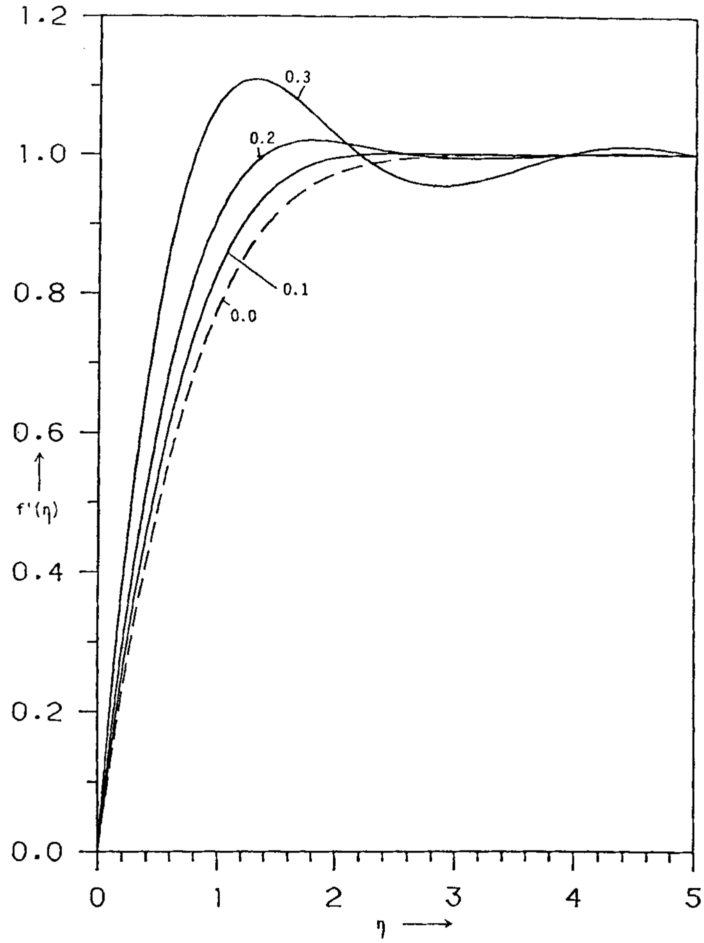

Figure 3. Variation of f' with η for various values of k. Flow near a stagnation point

FLOW OVER A STRETCHING SHEET

In this section we consider the flow of a Walters' B' fluid over a stretching sheet (see Figure 5). Here the motion is caused entirely by the stretching of the sheet. This problem is important in the polymer industry. The equation of motion for the problem is the same as equation (16) except that now $U=0$, which modifies equation (1) to

$$f'''+ff''-f'^2+k(ff^{iv}-2f'f'''+f''^2)=0. \tag{46}$$

The boundary conditions (22), because of the motion of the sheet, become

$$u=cx \quad \text{and} \quad v=0 \quad \text{at } y=0 \qquad u\rightarrow 0 \quad \text{as } y\rightarrow\infty, \tag{47}$$

which transform to

$$f(0)=0, \qquad f'(0)=1, \qquad \lim_{\eta\rightarrow\infty} f'(\eta)=0. \tag{48}$$

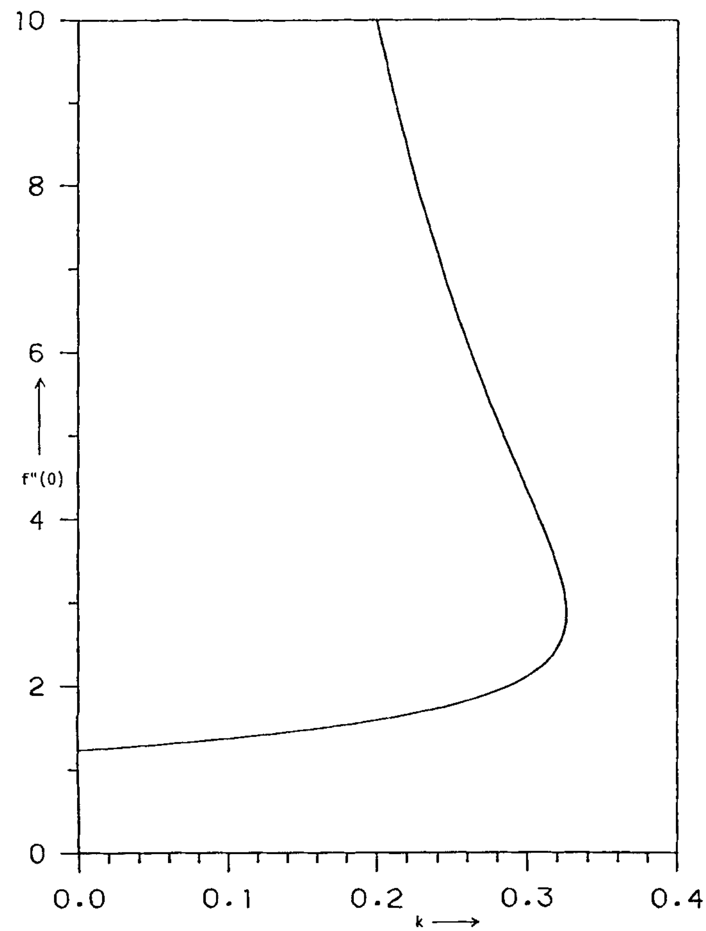

Figure 4. Variation of $f''(0)$ with k. Flow near a stagnation point

Rajagopal *et al.*[10] solved equation (46) by taking a regular perturbation expansion of f in k, retaining only the first-order terms. Its exact solution, found later by Troy *et al.*,[12] is

$$f(\eta)=(1-k)^{1/2}\{1-\exp[-\eta/(1-k)^{1/2}]\}. \tag{49}$$

Employing the method developed in the present paper, we shall solve equation (46) without making any approximation. The procedure for solution is very similar to that given for the flow near a stagnation point. We shall therefore only give the results here.

The value of k was increased from zero at intervals of 0·05 and the missing condition $f''(0)$ determined. In Figure 6, f is plotted against η for various values of k. In Figure 7, f' is plotted against η for the same set of values of k. One can conclude from these figures that as k increases, the velocity decreases, which effectively increases the boundary layer thickness. These results are qualitatively in agreement with those obtained by Rajagopal *et al.*[10] It would be of some interest to compare the value of τ, the dimensionless shear stress at the boundary given by

$$\tau=(1-k)f''(0), \tag{50}$$

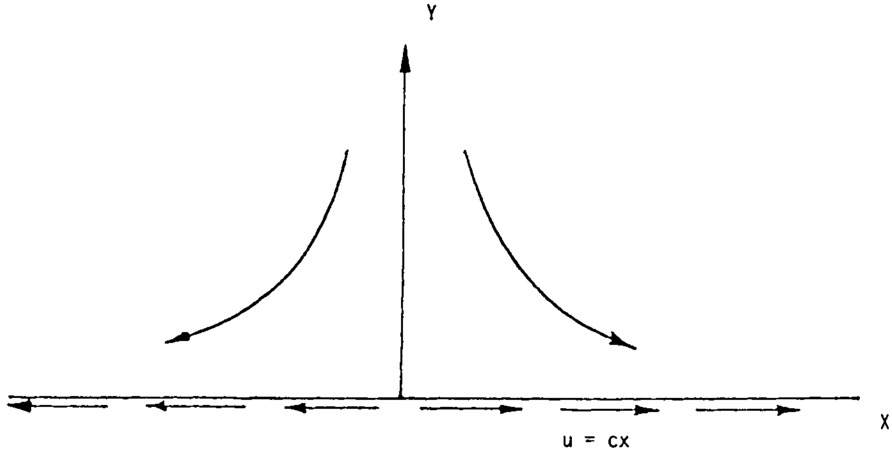

Figure 5. Geometry of flow over a stretching sheet

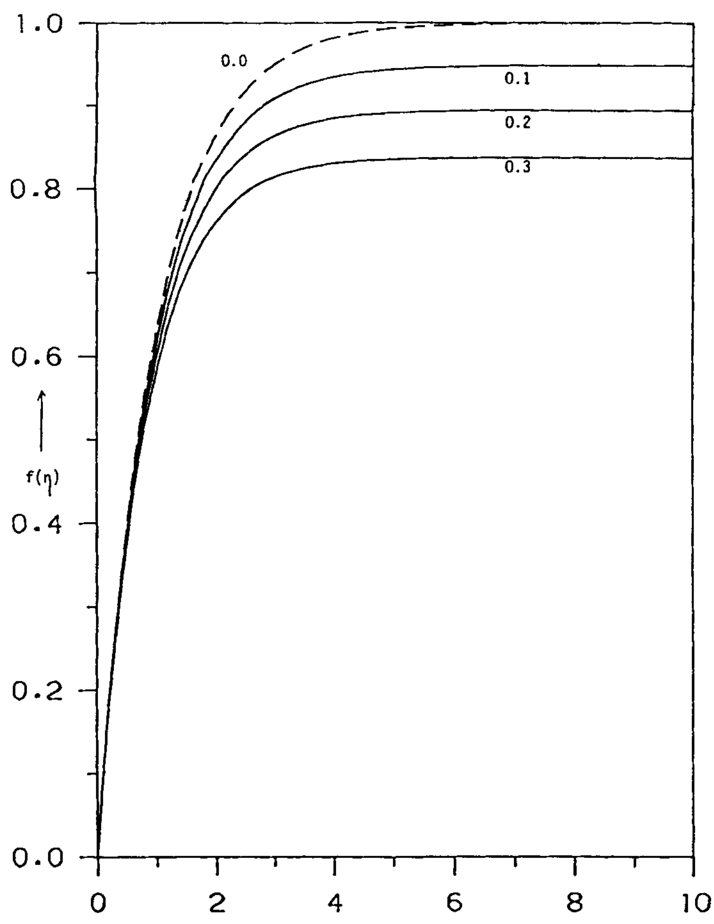

Figure 6. Variation of f with η for various values of k. Flow over a stretching sheet

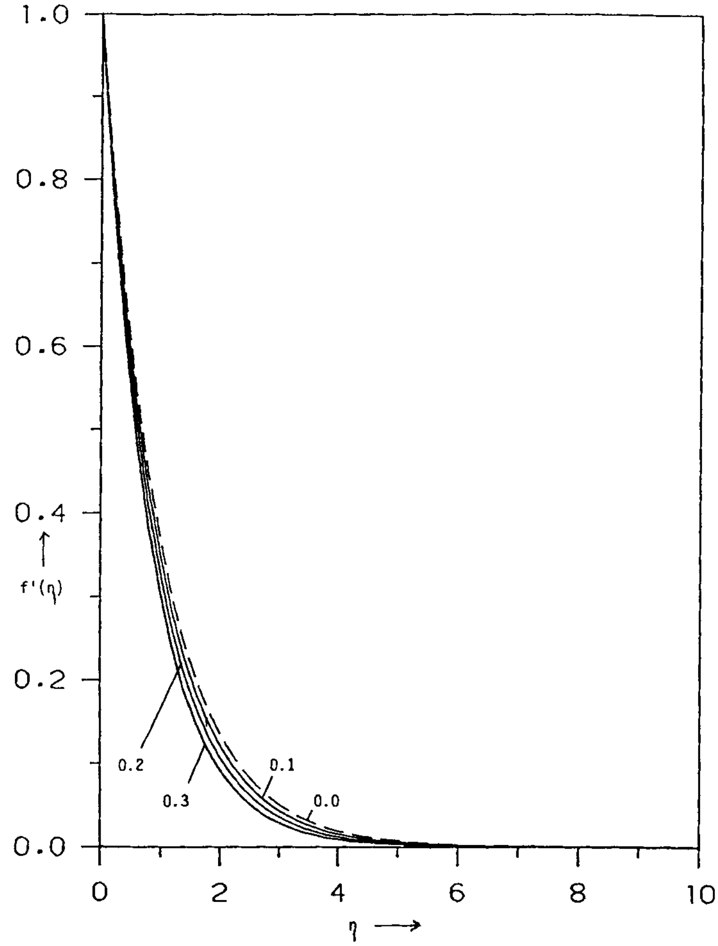

Figure 7. Variation of f' with η for various values of k. Flow over a stretching sheet

using the present method, the perturbation technique and the exact solution (49). These results are presented in Table I. For the values of k considered here, the values of τ obtained by the present method and the exact solution were in complete agreement until the last digit in the table. It can be further seen from the Table that the perturbation method systematically underestimates the values of τ with increasing values of k.

FLOW NEAR A ROTATING DISK

We now consider the flow of a Walters' B' fluid near a disk which is rotating about the z-axis with uniform velocity ω (Figure 8). Assuming steady rotational symmetric flow with the velocity components (v_r, v_θ, v_z), Elliot[11] obtained the following system of equations:

$$F''-HF'-F^2+G^2-k(HF'''+4FF''+2G'^2)=0, \quad (51)$$

$$G''-HG'-2FG-k(HG'''+4FG''-2F'G')=0, \quad (52)$$

Table I. Variation of τ with k for flow past a stretching sheet

k	τ Present method	τ Perturbation method
0	−1·0000	−1·0000
0·05	−0·9747	−0·9738
0·1	−0·9487	−0·9451
0·2	−0·8944	−0·8802
0·3	−0·8367	−0·8053

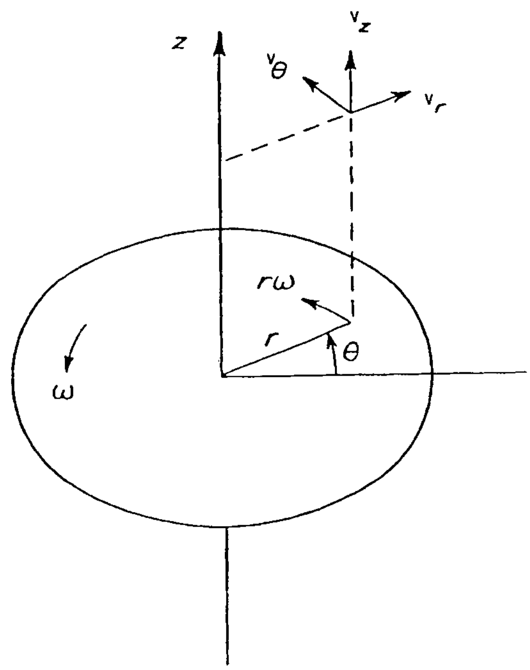

Figure 8. Geometry of flow near a rotating disk

$$2F+H'=0, \tag{53}$$

$$P'=H''-HH'+k(3H'H''-HH'''), \tag{54}$$

where

$$v_r=r\omega F(\zeta), \qquad v_\theta=r\omega G(\zeta), \qquad v_z=(v\omega)^{1/2}H(\zeta),$$
$$p=p(z)=sv\omega P(\zeta), \qquad k=k_0\omega/\eta_0. \tag{55}$$

Here ζ is the dimensionless distance from the disk defined by

$$\zeta=z(\omega/v)^{1/2}. \tag{56}$$

The boundary conditions of the problem are

$$v_r=0, \quad v_\theta=r\omega \quad \text{and} \quad v_z=0 \quad \text{at} \quad z=0, \qquad v_r\to 0 \quad \text{and} \quad v_\theta\to 0 \quad \text{as } z\to\infty, \tag{57}$$

which transform to

$$F(0)=0, \qquad H(0)=0, \qquad P(0)=0, \qquad G(0)=1, \tag{58}$$

$$F(\infty)=0, \qquad G(\infty)=0. \tag{59}$$

Following the standard procedure, Elliot[11] expanded F, G, H and P in powers of k, retaining only the first-order terms of k. Unfortunately, his equations for F_1, G_1, H_1 and P_1, the first-order perturbations, had a mistake in sign. The values for these quantities should have their signs reversed. We shall now solve the boundary value problem (BVP) defined by equations (51)–(54) with the boundary conditions (58) and (59) without using the perturbation technique.

Let

$$F'=S, \qquad G'=T; \tag{60}$$

then equations (51) and (52) become

$$S'-HS-F^2+G^2-k(HS''+4FS'+2T^2)=0, \tag{61}$$

$$T'-HT-2FG-k(HT''+4FT'-2ST)=0. \tag{62}$$

Discretizing equations (60)–(62) and (53), using the finite difference approximations

$$F'=\frac{F^{j+1}-F^{j-1}}{2h}, \qquad F''=\frac{F^{j+1}-2F^j+F^{j-1}}{h^2} \tag{63}$$

and the averages, we obtain

$$\frac{S^{j+1}-S^{j-1}}{2h}+H^jS^j-(F^j)^2+(G^j)^2-k\left(H^j\frac{S^{j+1}-2S^j+S^{j-1}}{h^2}+4F^j\frac{S^{j+1}-S^{j-1}}{2h}+2(T^j)^2\right)=0, \tag{64}$$

$$\frac{T^{j+1}-T^{j-1}}{2h}+H^jT^j-2F^jG^j-k\left(H^j\frac{T^{j+1}-2T^j+T^{j-1}}{h^2}+4F^j\frac{T^{j+1}-T^{j-1}}{2h}-2S^jT^j\right)=0, \tag{65}$$

$$F^{j+1}=F^j+\tfrac{1}{2}h(S^j+S^{j+1}), \tag{66}$$

$$G^{j+1}=G^j+\tfrac{1}{2}h(T^j+T^{j+1}), \tag{67}$$

$$H^{j+1}=H^j-h(F^j+F^{j+1}). \tag{68}$$

Equation (54) can be integrated directly to give the pressure P at any point.

The values of S^1 and T^1 can be obtained by expanding them in a Taylor series around $\eta=0$. We need to differentiate equations (51) and (52) to obtain $F'''(0)$ and $G'''(0)$. If

$$F'(0)=s, \qquad G'(0)=t, \tag{69}$$

we have

$$S^1=s+h(-1+2kt^2)-h^2(t+2ks), \tag{70}$$

$$T^1=t-2hkst+h^2[s+kt-2k^2t(s^2+t^2)], \tag{71}$$

the error in each of the above results being $O(h^3)$.

The integration can now be performed as follows. First some initial guess values are assigned to $F'(0)$ and $G'(0)$. S^1 and T^1 are then calculated using equations (70) and (71). F^1, G^1 and H^1 are next determined from equations (66)–(68). At the next cycle S^2 and T^2 are computed from equations (64) and (65). Again F^2, G^2 and H^2 are calculated from equations (66)–(68). As in the case of the flow near a stagnation point, the order in which the quantities S, T, F, G and H are calculated is important. The order indicated above is followed for the subsequent cycles. The integration is carried out until the values of the desired quantities are obtained at all the mesh points.

Note that we need to satisfy the two asymptotic boundary conditions (59). In fact s and t must be found by a shooting method so as to fulfil the boundary conditions (59). Here, amongst several choices, one can apply a variation of the secant method for two unknowns or use Newton's method. We chose the latter approach since it requires only one set of initial guess values. An accuracy of 10^{-10} was achieved after only five or six iterations. Once again Richardson's extrapolation was used to improve the accuracy of the results.

The flow was computed for values of k from zero to unity at intervals of 0·1. The most significant physical quantity of interest is the turning moment (or torque) for the disk with fluid on both sides. In dimensionless form it is given by

$$M = -G'(0). \tag{72}$$

The corrected result of Elliot[11] is

$$M = 0{\cdot}616 + 0{\cdot}208k, \tag{73}$$

which shows that the main effect of elasticity of type B' on a rotating disk is to increase the magnitude of the turning moment on the disk. This result appears to be in agreement with that derived by Rathna.[14] However, when the BVP was solved without making any approximation, i.e. using the present method, we got the results shown in Table II. We note from the table that M initially increases with k until it reaches its maximum value for $k = 0{\cdot}330\,97$. When k is further increased, M starts falling. It becomes less than the corresponding value for a Newtonian fluid for $k > 0{\cdot}734\,38$. For still larger values of k the value of the turning moment on the disk is less than that for a Newtonian fluid. This behaviour of M with k is shown in Figure 9.

Table II. Variation of M with k for flow near a rotating disk

k	M
0	0·61494
0·1	0·63314
0·2	0·64587
0·3	0·65168
0·4	0·65050
0·5	0·64358
0·6	0·63276
0·7	0·61971
0·8	0·60562
0·9	0·59125
1·0	0·57704

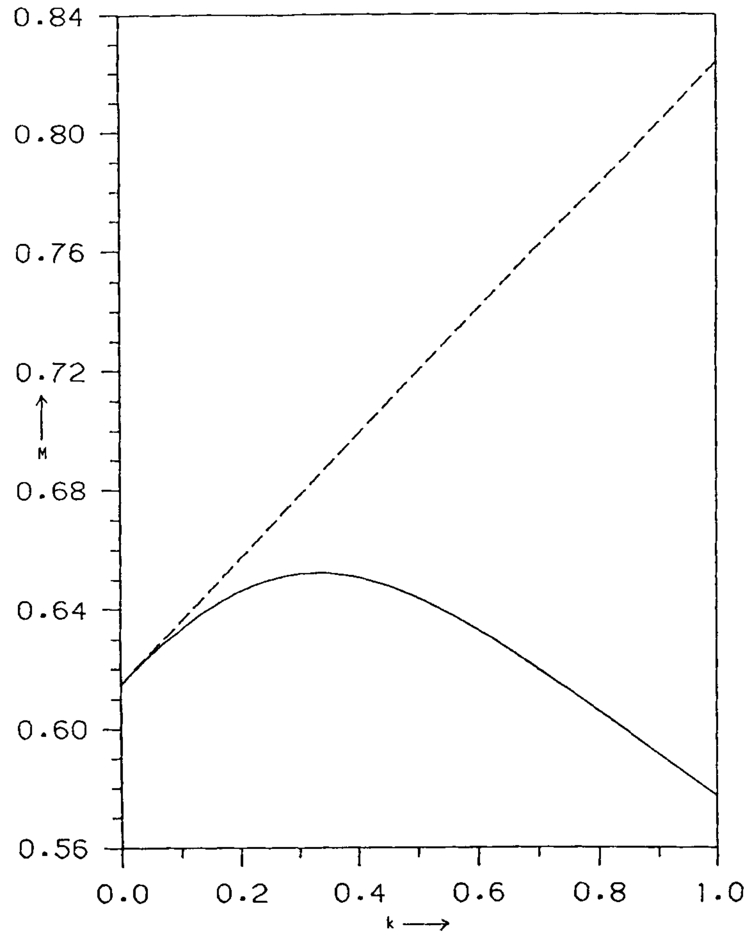

Figure 9. Variation of M, the turning moment of the disk, with k. Solid line denotes values obtained by present method. Dashed line denotes values obtained by perturbation method

CONCLUSIONS

In this paper we have presented a hybrid method for computing accurately the flows of viscoelastic fluids. The method combines the features of the finite difference technique and the shooting method. By using central differences for the derivatives, and averages, an accuracy $O(h^2)$ is preserved, which can be further enhanced to $O(h^4)$ by invoking Richardson's extrapolation. The shooting method is used to find the missing initial conditions, which can be systematically determined using any suitable zero-finding algorithm.

The present method operates on the full set of equations rather than on the perturbed sets. Moreover, it is applicable for all values of k, including $k=0$ and very small values of k. The application of the method can lead to somewhat unexpected results, as shown in the present paper for the problems of flow near a stagnation point and near a rotating disk. Using the perturbation technique, it would not have been easy to predict the existence of a turning point in the solution for the problem of flow near a stagnation point. Similarly, the increasing and then decreasing behaviour of the turning moment of a rotating disk with increasing k could not be predicted with ease using the perturbation technique.

It is hoped that the application of the present method will lead to many other interesting results for flow problems of viscoelastic fluids between porous channels, disks, etc. The results of these investigations will be reported in future communications.

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